Math 31 - Homework 4

Due Friday, July 20

Easy

1. Determine if each mapping is a homomorphism. State why or why not. If it is a homomorphism, find its kernel, and determine whether it is one-to-one and onto.

(a) Define $\varphi : \mathbb{Z} \to \mathbb{R}$ by $\varphi(n) = n$. (Both are groups under addition here.)

(b) Let G be a group, and define $\varphi : G \to G$ by $\varphi(a) = a^{-1}$ for all $a \in G$.

(c) Let G be an *abelian* group, and define $\varphi : G \to G$ by $\varphi(a) = a^{-1}$ for all $a \in G$.

(d) Define $\varphi : \mathrm{GL_n}(\mathbb{R}) \to \mathbb{R}^\times$ by $\varphi(A) = \det(A)$. (Recall that $\mathbb{R}^\times$ denotes the nonzero real numbers under multiplication.)

(e) Let G be a group, and define $\varphi : G \to G$ by $\varphi(a) = a^2$ for all $a \in G$.

2. Consider the subgroup $H = \{i, m_1\}$ of the dihedral group D_3. Find all the left cosets of H, and then find all of the right cosets of H. Observe that the left and right cosets do not coincide.

3. Let G and G' be groups, and suppose that $|G| = p$ for some prime number p. Show that any group homomorphism $\varphi : G \to G'$ must either be the trivial homomorphism or a one-to-one homomorphism.

Medium

4. Let G be a group and H a subgroup of G. Show that there are the same number of left cosets of H as there are right cosets of H. That is, exhibit a one-to-one map from the set of all left cosets of H onto the set of all right cosets of H. (Note that this can be accomplished for finite groups by simply counting. Your proof must work for *all* groups.)

5. [Herstein, Section 2.5 #2] Recall that $G_1 \cong G_2$ means that G_1 is isomorphic to G_2. Prove the following statements.

(a) For any group G, we have $G \cong G$.

(b) If G_1 and G_2 are groups and $G_1 \cong G_2$, then $G_2 \cong G_1$.

(c) If G_1, G_2, and G_3 are groups, and $G_1 \cong G_2$ and $G_2 \cong G_3$, then $G_1 \cong G_3$.

[Note that you are essentially proving that isomorphism is an equivalence relation on the class of all groups.]

6. [Herstein, Section 2.5 #14] If G is abelian and $\varphi : G \to G'$ is a homomorphism from G *onto* G', prove that G' is abelian.

Hard

7. [Herstein, Section 2.5 #28] Let G be a group, and let $\mathrm{Aut}(G)$ denote the set of all automorphisms of G. We can define a binary operation on $\mathrm{Aut}(G)$ by:

$$\theta\psi = \theta \circ \psi$$

for $\theta, \psi \in \mathrm{Aut}(G)$.

(a) Prove that if $\theta, \psi \in \mathrm{Aut}(G)$, then $\theta\psi \in \mathrm{Aut}(G)$. (That is, show that we have indeed defined a binary operation by checking that $\mathrm{Aut}(G)$ is closed.

(b) If $\theta \in \mathrm{Aut}(G)$, then θ is in particular a bijection, so it has an inverse θ^{-1}. Prove that θ^{-1} is a homomorphism, so that $\theta^{-1} \in \mathrm{Aut}(G)$ for all $\theta \in \mathrm{Aut}(G)$.

(c) Use parts (a) and (b) to show that $\mathrm{Aut}(G)$ is itself a group under composition.